MATH2050C Assignment 10

Deadline: April 1, 2025.

Hand in: 5.3 no. 1, 5, 6, 12, 15. Suppl Problem no 1.

Section 5.3 no. 1, 3, 4, 5, 6, 12, 13, 15, 17.

Supplementary Problems

1. Let $f \in C[a, b]$. Suppose that $f(x) > 0$ for all $x \in [a, b]$. Show that there is some $\rho > 0$ such that $f(x) \geq \rho$ for all $x \in [a, b]$. Hint: Use Suppl. Problem no 3 in Ex 9 and apply the Heine-Borel Theorem.

2. Use the previous problem to deduce the Max-Min Theorem.

3. Let A be a set in $\mathbb{R}$ which contains at least two points and satisfies: Whenever $x, y \in A, x < y$, implies the interval $[x, y] \subset A$. Prove that A is an interval.

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Heine-Borel Theorem

A collection of open intervals $\{I_\alpha\}, \alpha \in A$, is called an open covering of the set A if $A \subset \bigcup_{\alpha\in A} I_\alpha$. Here A could be an uncountable index set.

Theorem 10.1 (Heine-Borel Theorem) Let $\{I_\alpha\}$ be an open covering of the interval $[a,b]$. There are finitely many $I_{\alpha_1}, \cdots, I_{\alpha_n}$ so that $[a,b] \subset \bigcup_{j=1,\cdots,n} I_{\alpha_j}$.

In other words, every open cover of $[a,b]$ admits a finite subcover.

Proof: Suppose on the contrary there is no finite subcover of $[a,b]$. Divide $[a,b]$ equally into two subintervals. Then at least one of these subintervals does not admit any finite subcover. Pick one and call it J_1. Dividing J_1 equally and repeating our selection, in this way we obtain a sequence of closed intervals $\{J_j\}, J_{j+1} \subset J_j$ with length decreasing to 0. Each J_j does not admit any finite subcover from $\{I_\alpha\}$. By Nested Interval Theorem, there is $z \in \bigcap_j J_j$. Since $z \in [a,b]$ and $\{I_\alpha\}$ forms a cover of $[a,b]$, $z \in I_{\alpha_0}$ for some I_{α_0}. It is clear that for sufficiently large j, $J_j \subset I_{\alpha_0}$. But then J_j is covered by the single I_{α_0}, contradicting the assumption that J_j does not admit any finite subcover from $\{I_\alpha\}$.

As an illustration of the applications of this theorem, we present an alternative proof of the boundedness theorem.

Theorem 10.2 (Boundedness Theorem) Every continuous function on $[a,b]$ is bounded.

Proof: For $x \in [a,b]$, we claim there are some $\delta_x > 0$ and $M_x > 0$ such that $|f(y)| \le M_x,\ y \in (x-\delta_x, x+\delta_x)$. (We have put an index x under M and δ to emphasis their dependence on x.) Well, just take $\varepsilon = 1$, there is some δ_x such that $|f(y) - f(x)| < 1$ for $y, |y-x| < \delta_x$. It follows that

$$|f(y)| \le |f(y) - f(x)| + |f(x)| \le 1 + f(x) \equiv M_x \ , \quad y \in (x-\delta_x, x+\delta_x).$$

The open intervals $I_x = (x-\delta_x, x+\delta_x)$ forms an open cover of $[a,b]$. By Heine-Boral Theorem there is a finite subcover by $I_{x_1}, \cdots, I_{x_n}$ so that $[a,b] \subset \bigcup_i I_{x_i}$. Every $x \in [a,b]$ must belong to some I_{x_i}, so $|f(x)| \le M \equiv \max\{M_{x_1}, \cdots, M_{x_n}\}$.

Continuous Functions in $\mathbb{R}^n$

A sequence $\{p_n\}$ in $\mathbb{R}^N$ is called convergent to p if for each $\varepsilon > 0$, there is some n_0 such that $|p_n - p| < \varepsilon$ for all $n \ge n_0$. For $p \in \mathbb{R}^n$, $|p| = (\sum_{k=1}^N x_k^2)^{1/2}, p = (x_1, \cdots, x_N)$ is its Euclidean norm and $|p_n - p|$ is the Euclidean distance between p_n and p. It is easy to see that $p_n \to p$ iff $x_1^n \to x_1, \cdots, x_n^N \to x_N$.

Let f be defined on $A \subset \mathbb{R}^N$ and $x \in A$. It is continuous at p if for each $\varepsilon > 0$, there is some δ such that $|f(q) - f(p)| < \varepsilon$ for all $q \in A, |q-p| < \delta$.

A nonempty set in $\mathbb{R}^N$ is closed if it contains all its cluster points. In other words, A is closed if whenever $\{p_n\} \subset A, p_n \to p$ implies that $p \in A$. We will use a closed, bounded set to replace a closed, bounded interval in higher dimensions. The following theorems can be proved as in the one dimensional case by the Bolzano-Weierstrass Theorem whose higher dimension version is easy to obtain.

Theorem 10.3 (Boundedness Theorem) Let f be continuous on a closed, bounded set K in $\mathbb{R}^N$. There exists some M such that $|f(p)| \le M$ for all $p \in K$.

Theorem 10.4 (Max-Min Theorem) Let f be continuous on a closed, bounded set K in $\mathbb{R}^N$. There exist p_1 and p_2 in K such that $f(p_1) \le f(p)$ and $f(p) \le f(p_2)$ for all $p \in K$.

The extension of the Location of Root Theorem to $\mathbb{R}^N$ requires the notion of connectedness.

A set E in $\mathbb{R}^n$ is connected if every two points p and q in E can be connected by a continuous curve from p to q in E, that is, $\gamma(0) = p, \gamma(1) = q$. Recall from Advanced Calculus that a continuous curve $\gamma(t) = (\gamma_1(t), \cdots, \gamma_n(t))$ is map from $[0,1]$ to E where all components γ_k's are continuous.

Theorem 10.5 (Root Theorem) Let f be continuous on a connected set E in $\mathbb{R}^N$. Suppose there are p and q in E satisfying $f(p)f(q) < 0$. There is some $r \in E$ such that $f(r) = 0$.

The proof is a reduction to one-dimension. Fix a continuous curve γ in E connecting p to q. Consider the composite function $g(t) = f(\gamma(t))$ which is a continuous function on $[0,1]$ to $\mathbb{R}$. As $g(0)g(1) = f(p)f(q) < 0$, By Root Theorem, $g(c) = 0$ for some $c \in (0,1)$. Thus $f(r) = 0$ where $r = \gamma(c)$.

Remark In Advanced Calculus we study integration on regions. A region consists of all points lying inside a closed curve (or bounded by several closed curves) as well as all the boundary points (that is, points on these curves). It is a closed, bounded, connected set.